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Determinisation of relational substitutions in ordered categories with domain ☆

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ABSTRACT

Restating a unification problem as a single relational substitution instead of as multiple functional substitutions (or terms), a solution becomes a "determiniser" arrow and allows formalisation in the context of locally ordered categories with domain. This relies on the determinacy concept of "characterisation by domain" introduced by Desharnais and Möller for Kleene algebras with domain; this is here applied in the weakest possible setting.

We show how "most general determinisers" can be seen as generalisation of quotient projections of partial equivalence relations, and show a characterisation that manages to avoid using converse or symmetry by employing restricted residuals instead.

1. Introduction

Substitutions have been considered in a categorical context at least since the seminal work by Lawvere [27]. In that context, a unification problem can be stated as a pair of parallel arrows, and their most general unifier is then just their co-equaliser.

Relational substitution concepts allow more liberal ways to formulate unification problems, in particular as a *single*, relational morphism. In [23], we considered two different categories for "relational" concepts of substitutions:

- "Relational substitutions" can be understood as non-deterministic variable bindings, and have a composition that corresponds to call-by-name, or "run-time choice".
- "Substitution sets" correspond to non-deterministic choices of standard substitutions, and therefore more closely correspond to call-by-value, or "call-time choice".

The greatest common denominator of these two relational substitution concepts is the setting of ordered categories with domain. In this setting, the essence of being "unified" can be captured via the determinacy concept of domain minimality, introduced by Desharnais and Möller [10]. We therefore replace the co-equaliser-based definition of unifiers with a new definition of "determiniser". In [23], we showed that *initial* determinisers relate usefully to relational translations of conventional substitution problems in both relational substitution concepts.

However, the initiality condition, a typical category-theoretic universal property, is rather unsatisfying in a relational context. Relational characterisations corresponding to such universal properties are typically local, i.e., involve no quantification over objects, and rarely quantification over morphisms. Typical examples are relational characterisations of direct products [5], called tabulations by Freyd and Scedrov [18].

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We identify the quotient characterisation as the one closest to the determiniser concept. However, since our locally ordered categories of relational substitutions do not provide symmetry, we have to find other ways to deal with what normally are the symmetric aspects of the quotient characterisation.

Finally, the decomposition of the unification problem along the subterm structure requires a further adaptation, based on an abstract representation of term positions in terms of membership of a monad functor.

We assemble the theoretical foundations of our discussion in three strands:

- The basic theory of ordered categories with domain (Section 2) can serve as common stratum of more complete relation-algebraic theories. We use these theories to reason about relational substitutions in Sections 4 and 5, where we concentrate on properties of relational substitutions that can be expressed in the basic setting of ordered categories with domain (and range).
- Monads are an abstraction that is particularly useful for container datatypes, like terms; we collect the necessary definitions in Section 3 and show relevant properties of the Kleisli category. A related concept in that context is that of membership of datatypes, which is again relevant for terms; we present this in Section 8, and show also how membership is useful in the Kleisli category, and that the term monad has membership.
- Right residuals approximate matching, and left residuals are one way to obtain the kernel of a function; restricted variants of the conventional residuals deal more conveniently with partiality and lack of surjectivity (Section 7).

Section 5 also lists some properties that hold automatically for relational substitutions, and presents counterexamples for some closely related properties that do not hold. Section 6 contains the central determiniser definitions. The central part of our search for appropriate characterisations of the determiniser concept in Section 9 starts from the much simpler situation in appropriate allegories [18], i.e., categories of relations with converse, and strives to derive converse-free formalisations that can also be used in ordered categories of substitutions, which do not have converse. The problem of precise characterisation of determinisers for relational substitutions is then treated in Section 10. Finally, in Section 11, we discuss some related work.

2. Ordered categories with domain

In our use of category theoretical concepts, we write composition using the “diagrammatic” convention:

Notation 2.1. In a *category*, we write $\mathbb{I}_{\mathcal{A}}$ for the identity on object $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ to say that F is a morphism from object $\mathcal{A}$ to object $\mathcal{B}$. The *homset* of all morphisms from $\mathcal{A}$ to $\mathcal{B}$ is also written $\mathsf{Hom}(\mathcal{A}, \mathcal{B})$.

For two morphisms $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, we write $F ; G$ for their composition; we then have $(F ; G) : \mathcal{A} \to \mathcal{C}$. □

One basic additional feature of categories of relations is the inclusion ordering among relations between the same two sets; this motivates our use of locally ordered categories:

Definition 2.2. An *(locally) ordered category* is a category in which on each homset $\mathsf{Hom}(\mathcal{A}, \mathcal{B})$, there is an ordering[1] $\subseteq_{\mathcal{A},\mathcal{B}}$, and composition is monotonic in both arguments. □

We will normally omit the subscripts, as they can be deduced from the context.

Definition 2.3. In an ordered category, we call a morphism $R : \mathcal{A} \to \mathcal{A}$

- *reflexive* iff $\mathbb{I}_{\mathcal{A}} \subseteq R$,
- *subidentity* iff $R \subseteq \mathbb{I}_{\mathcal{A}}$,
- *transitive* iff $R ; R \subseteq R$,
- *idempotent* iff $R ; R = R$. □

For special cases of the local ordering we recall (e.g. from Kahl [22]):

Definition 2.4. An ordered category is called

- a *lower semilattice category* if each homset has binary meets,
- a *upper semilattice category* if each homset has binary joins, and composition distributes over these,
- a *complete upper semilattice category* if each homset has arbitrary joins, and composition distributes over these,
- *having zero morphisms*, if each homset has a least element (which is the join of the empty set), and these behave as zeros (which is distribution over the empty join).

A *Kleene category* is an upper semilattice category with zero morphisms where on homsets of endomorphisms there is an additional unary operation $\_^*$ such that $R^* = \mathbb{I}_A \cup R \cup R^* ; R^*$ and the induction laws hold:

$$Q ; R \subseteq Q \Rightarrow Q ; R^* \subseteq Q \qquad \text{and} \qquad R ; S \subseteq S \Rightarrow R^* ; S \subseteq S$$

[1] An ordering is a reflexive, transitive, and antisymmetric relation.

In a Kleene category, we also define $R^+ := R^* ; R$. □

A complete upper semilattice category is automatically a Kleene category. In the Kleene category Rel of binary relations between sets, R^+ is the transitive closure of R and R^* is the reflexive and transitive closure of R.

If objects are seen as corresponding to sets, *idempotent subidentities* correspond to subsets. In the Kleene algebra context, such subidentities are called "tests" or "domain elements"; Backhouse calls them "monotypes", e.g. in [12]. In the context of unification, the dual concept of range is perhaps even more important than domain, but we follow the customary approach of introducing domain first, and adapt the domain definition of Desharnais et al. [11] to the setting of ordered categories:

Definition 2.5. An *ordered category with pre-domain* is an ordered category where for every morphism $R : \mathcal{A} \to \mathcal{B}$ there is an idempotent subidentity $\mathsf{dom}\, R : \mathcal{A} \to \mathcal{A}$ such that

- for every morphism $S : \mathcal{A} \to \mathcal{B}$ with $R \supseteq S$ we have $(\mathsf{dom}\, R) ; S \supseteq S$, and
- for every idempotent subidentity $q : \mathcal{A} \to \mathcal{A}$ with $q ; R \supseteq R$, we have $q \supseteq \mathsf{dom}\, R$.

In an *ordered category with domain*, additionally the following *locality* condition holds:

$$\mathsf{dom}\,(R ; \mathsf{dom}\, S) \subseteq \mathsf{dom}\,(R ; S) \qquad (D2 \subseteq)$$

Range $\mathsf{ran}\, R : \mathcal{B} \to \mathcal{B}$ is defined dually.

Prefix operators like dom and ran have higher precedence than binary operators, so $\mathsf{dom}\, R ; S = (\mathsf{dom}\, R) ; S$. □

Already a pre-domain operator is monotonic and idempotent, preserves idempotent subidentities, and satisfies the opposite inclusion to $(D2 \subseteq)$, and

$$\mathsf{dom}\, R ; R = R \ . \qquad (D1)$$

A domain operator therefore in addition satisfies in particular Eq. $(D2)$, and also the following:

$$\mathsf{dom}\,(\mathsf{dom}\, R ; S) = \mathsf{dom}\, R ; \mathsf{dom}\, S \qquad (D3) \qquad \mathsf{dom}\, R ; \mathsf{dom}\, S = \mathsf{dom}\, S ; \mathsf{dom}\, R \qquad (D4)$$

Desharnais et al. [9] list Eqs. $(D1)$–$(D4)$ as the axioms for domain semigroups in a context where no local order is given *a priori*.

In allegory and relation algebra contexts, many properties are normally defined using converse; some of these can be defined using domain instead:

Definition 2.6. In an ordered category with domain (respectively, range), we call a morphism $R : \mathcal{A} \to \mathcal{B}$

- *total* iff $\mathsf{dom}\, R = \mathbb{I}_{\mathcal{A}}$,
- *surjective* iff $\mathsf{ran}\, R = \mathbb{I}_{\mathcal{B}}$. □

For the property of univalence, usually defined using converse as $R^{\smile} ; R \subseteq \mathbb{I}$, it is harder to find an appropriate replacement that does not use converse; Desharnais and Möller [10] have studied this problem extensively; we will mainly use the property they introduced as "characterisation by domain (CD)":

Definition 2.7. In an ordered category with domain, a morphism $F : \mathcal{A} \to \mathcal{B}$ is called *deterministic* iff F is *domain-minimal*, i.e., iff[2]

$$\forall R : \mathcal{A} \to \mathcal{B} \bullet \quad R \subseteq F \quad \Rightarrow \quad R = \mathsf{dom}\, R ; F. \qquad \square$$

[2] For quantification and set comprehension we shall use the notation of Z [33], which uses the pattern

$\forall$ *declaration* | *range-predicate* • *predicate*

to denote the truth of *predicate* for all bindings of the variables in *declaration* that satisfy the *range-predicate*, respectively,

{ *declaration* | *range-predicate* • *term* }

to denote the set of all values of *term* under bindings for the locally bound variables from *declaration* that satisfy the *range-predicate* (which defaults to True), for example:

$$\forall z : \mathbb{Z} \mid z < -5 \bullet z^2 > 30 \qquad \forall k : \mathbb{N} \bullet k + 2 > 0 \qquad \{k : \mathbb{N} \mid k < 4 \bullet k^2\} = \{0, 1, 4, 9\} \qquad \{n : \mathbb{N} \bullet (-1)^n\} = \{-1, 1\}$$

Interestingly, an identity $\mathbb{I}_\mathcal{A}$ is only deterministic if for each subidentity p on $\mathcal{A}$ we have $\mathrm{dom}\, p = p$, or equivalently, that p is idempotent.

Definition 2.8. In an ordered category with domain, a morphism $F : \mathcal{A} \to \mathcal{B}$ is called a *mapping* iff F is deterministic (domain-minimal) and total. □

While we use idempotent subidentities for the purpose of characterising "parts" of objects, another approach common in the full relation algebra context is to use vectors, i.e., relations with $v = v \,;\, \top$. In this context, Schmidt and Ströehlein [32] proposed to use injective vectors called "points" to correspond to single elements. In the context of ordered categories with domain, but not necessarily with top morphisms $\top$, the point concept takes on the following shape:

Definition 2.9. In an ordered category with domain, we call an idempotent subidentity $p : \mathcal{A} \to \mathcal{A}$ a *point* iff for all $R : \mathcal{A} \to \mathcal{A}$, the composition $R \,;\, p$ is deterministic. □

3. Ordered monads

Functor and monad concepts are easily transferred to the setting of ordered categories — relators as "relational functors" have originally been introduced by Kawahara [25]; Definitions 3.1, 3.3, and 3.4 are adapted to our setting from Backhouse [2]:

Definition 3.1. A *relator* between two ordered categories is a monotonic functor. □

Lemma 3.2. *If $\mathcal{F}$ is a relator from* **C** *to* **D***, both ordered categories with domain, and $R : \mathcal{A} \to \mathcal{B}$ in* **C***, then*

$$\mathsf{dom}_{\mathbf{D}}\, (\mathcal{F}\, R) \subseteq \mathcal{F}\, (\mathsf{dom}_{\mathbf{C}}\, R)$$

Proof. Since $\mathcal{F}$ is a relator, $\mathcal{F}\, (\mathsf{dom}_{\mathbf{C}}\, R)$ is an idempotent subidentity, and the statement follows via the domain definition from $\mathcal{F}\, (\mathsf{dom}_{\mathbf{C}}\, R) \,;\, (\mathcal{F}\, R) = \mathcal{F}\, (\mathsf{dom}_{\mathbf{C}}\, R \,;\, R) = \mathcal{F}\, R$. □

The converse inclusion holds in allegories, but not in general.

For relators, we frequently require natural transformations that are restricted to consist only of mappings:

Definition 3.3. A *natural simulation* τ from a relator $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ to a relator $\mathcal{G} : \mathbf{C} \to \mathbf{D}$ is a family of total and deterministic morphisms in **D** (which therefore needs domain) indexed with objects of **C** such that $\tau_\mathcal{A} : \mathcal{F}\,\mathcal{A} \to \mathcal{G}\,\mathcal{A}$ and for every $R : \mathcal{A} \to \mathcal{B}$ we have $\mathcal{F}\, R \,;\, \tau_\mathcal{B} = \tau_\mathcal{A} \,;\, \mathcal{G}\, R$. □

Employing these restricted natural simulations is the only adaptation necessary to obtain a variant of the standard monad definition that is useful for monads in particular over ordered categories of relations:

Definition 3.4. An *ordered monad* is a triple $(\mathcal{M}, \eta, \mu)$ such that $\mathcal{M} : \mathbf{C} \to \mathbf{C}$ is an endo-relator, and $\eta : \mathbb{I} \to \mathcal{M}$ and $\mu : \mathcal{M} \,;\, \mathcal{M} \to \mathcal{M}$ are natural simulations satisfying associativity: $\mu_{\mathcal{M}\,\mathcal{A}} \,;\, \mu_\mathcal{A} = \mathcal{M}\, \mu_\mathcal{A} \,;\, \mu_\mathcal{A}$ and the unit laws: $\eta_{\mathcal{M}\,\mathcal{A}} \,;\, \mu_\mathcal{A} = \mathbb{I}_{\mathcal{M}\,\mathcal{A}}$ and $\mathcal{M}\, \eta_\mathcal{A} \,;\, \mu_\mathcal{A} = \mathbb{I}_{\mathcal{M}\,\mathcal{A}}$. □

For such an ordered monad over **C**, the Kleisli category $\mathbb{K}\,\mathcal{M}$ is defined as usual, with η for identities, and composition of $R : \mathcal{A} \to \mathcal{M}\,\mathcal{B}$ and $S : \mathcal{B} \to \mathcal{M}\,\mathcal{C}$ defined as $R \fatsemi S := R \,;\, \mathcal{M}\, S \,;\, \mu_\mathcal{C}$.

Monotonicity of composition in the Kleisli category (which inherits the ordering from **C**) follows from monotonicity of composition in **C** together with monotonicity of $\mathcal{M}$.

Since $\eta_\mathcal{A}$ is deterministic in **C**, having $R \subseteq \eta_\mathcal{A}$ implies $R = \mathsf{dom}_{\mathbf{C}}\, R \,;\, \eta_\mathcal{A}$, so the subidentities in the Kleisli category are all idempotent, are also in one-to-one correspondence with the idempotent subidentities in **C**, which produces a domain operation in the Kleisli category:

Lemma 3.5. *The Kleisli category for an ordered monad $(\mathcal{M}, \eta, \mu)$ over an ordered category* **C** *with domain is an ordered category with pre-domain, and* $\mathsf{dom}_{\mathbb{K}\,\mathcal{M}}\, R = (\mathsf{dom}_{\mathbf{C}}\, R) \,;\, \eta$.

If $\mathcal{M}\, (\mathsf{dom}_{\mathbf{C}}\, R) \subseteq \mathsf{dom}_{\mathbf{C}}\, (\mathcal{M}\, R)$ for all R, then locality is satisfied, too, and the Kleisli category then is an ordered category with domain. □

This extends to determinism:

Lemma 3.6. *Furthermore, $S : \mathcal{A} \to \mathcal{M}\,\mathcal{B}$ is domain-minimal in $\mathbb{K}\,\mathcal{M}$ iff it is domain-minimal in* **C**.

Proof. We show the last statement using the domain equation:

	S is domain-minimal in $\mathbb{K}\,\mathcal{M}$	
$\Leftrightarrow$	$\forall R : \mathcal{A} \to \mathcal{M}\,\mathcal{B} \bullet R \subseteq S \Rightarrow R = \mathsf{dom}_{\mathbb{K}\,\mathcal{M}}\, R \,⨾\, S$	Def. domain-minimal
$\Leftrightarrow$	$\forall R : \mathcal{A} \to \mathcal{M}\,\mathcal{B} \bullet R \subseteq S \Rightarrow R = \mathsf{dom}_{\mathbf{C}}\, R \,;\, \eta_{\mathcal{A}} \,;\, \mathcal{M}\,S \,;\, \mu_{\mathcal{B}}$	Lemma 3.5, Def. Kleisli composition
$\Leftrightarrow$	$\forall R : \mathcal{A} \to \mathcal{M}\,\mathcal{B} \bullet R \subseteq S \Rightarrow R = \mathsf{dom}_{\mathbf{C}}\, R \,;\, S \,;\, \eta_{\mathcal{M}\,\mathcal{B}} \,;\, \mu_{\mathcal{B}}$	Naturality of η
$\Leftrightarrow$	$\forall R : \mathcal{A} \to \mathcal{M}\,\mathcal{B} \bullet R \subseteq S \Rightarrow R = \mathsf{dom}_{\mathbf{C}}\, R \,;\, S$	Unit law
$\Leftrightarrow$	S is domain-minimal in $\mathbf{C}$	Def. domain-minimal □

From the definition of composition in the Kleisli category we easily obtain one half of join preservation — this could also be stated in terms of the "lazy" or "left" Kleene algebras of Möller [28]:

Lemma 3.7. *If* $\mathbf{C}$ *is a (complete) upper semilattice category (with zero morphisms), then composition in the Kleisli category distributes over binary (and arbitrary) (and empty) joins to its left.*

Proof. With join-distributivity in $\mathbf{C}$, we have (for a two-element, respectively, arbitrary, respectively, empty set $\mathcal{S}$):

$$(\bigcup \mathcal{S}) ⨾ T = (\bigcup \mathcal{S}) \,;\, \mathcal{M}\,T \,;\, \mu_{\mathcal{C}} = \bigcup \{S : \mathcal{S} \bullet S \,;\, \mathcal{M}\,T \,;\, \mu_{\mathcal{C}}\} = \bigcup \{S : \mathcal{S} \bullet S ⨾ T\}$$ □

Preservation of different kinds of joins in the right argument of composition additionally requires preservation of these joins by the relator $\mathcal{M}$:

Lemma 3.8. *If* $\mathbf{C}$ *is a (complete) upper semilattice category (with zero morphisms) and the monad functor* $\mathcal{M}$ *preserves binary (and arbitrary) (and empty) joins, then the Kleisli category is a (complete) upper semilattice category (with zero morphisms) again.* □

4. Signatures and terms

For the sake of minimising notational overhead for the motivating example, we only consider single-sorted signatures. Also, since we do not need to distinguish constant symbols from zero-ary function symbols, we allow arbitrary natural numbers as arities of function symbols and do not consider separate constant symbols.

Definition 4.1. A *signature* $\Sigma = (\mathcal{F}, \mathsf{arity})$ consists of a set $\mathcal{F}$ of *function symbols* and a total mapping $\mathsf{arity} : \mathcal{F} \to \mathbb{N}$ assigning each function symbol the number of arguments it requires in term construction.

A signature is called *unary* if it contains only unary function symbols. □

Given the ordered category of sets and relations as base, a signature $\Sigma = (\mathcal{F}, \mathsf{arity})$ can be considered as a relator mapping each set $\mathcal{S}$ to the set of function symbol applications, i.e., constructs $f(x_1, \ldots, x_{\mathsf{arity}\,f})$ for a function symbol $f \in \mathcal{F}$ and elements $x_1, \ldots, x_{\mathsf{arity}\,f} \in \mathcal{S}$, and having $\Sigma\,R$ relate two terms $f(t_1, \ldots, t_{\mathsf{arity}\,f})$ and $g(u_1, \ldots, u_{\mathsf{arity}\,g})$ if and only if $f = g$ and R relates t_i to u_i for each $i \in \{1, \ldots, \mathsf{arity}\,f\}$.

The term relator $\mathcal{T}_\Sigma$ is then the initial solution of the following equation in $\mathcal{T}_\Sigma$ (with a direct sum bifunctor "+"):

$$\mathcal{T}_\Sigma\, X = X + \Sigma\,(\mathcal{T}_\Sigma\, X)$$

We could gain considerable rigour by using the full formal machinery including catamorphisms for inductive datatypes as presented e.g. by Backhouse and Hoogendijk [1] or by Bird and de Moor [8]. However, this would require significant additional formalism to be introduced, the costs of which appear to outweigh its benefits, in particular since concrete substitutions arising from the term relator serve mainly as motivation for the more abstract development in the current paper.

We elide the isomorphism that usually connects the two sides of the term relator equation, and we use the following notation for the injections of the sum $\mathcal{T}_\Sigma\, X = X + \Sigma\,(\mathcal{T}_\Sigma\, X)$:

$\mathbb{V}_{\Sigma,\mathcal{X}} : \mathcal{X} \rightarrowtail \mathcal{T}_\Sigma\, \mathcal{X}$		"**v**ariable injection"
$\mathbb{A}_{\Sigma,\mathcal{X}} : \Sigma\,(\mathcal{T}_\Sigma\, \mathcal{X}) \rightarrowtail \mathcal{T}_\Sigma\, \mathcal{X}$		"symbol **a**pplication"

We also use the notation $\langle R, S \rangle$ to compose a morphism $\langle R, S \rangle : X + \Sigma\,(\mathcal{T}_\Sigma\, X) \to C$ from two morphisms $R : X \to C$ and $S : \Sigma\,(\mathcal{T}_\Sigma\, X) \to C$.

The "free extension" (see e.g. [19]) $\mathbb{E}_{\Sigma,\mathcal{X}} : \mathcal{T}_\Sigma(\mathcal{T}_\Sigma\,\mathcal{X}) \to \mathcal{T}_\Sigma\,\mathcal{X}$ maps "terms over terms" to "terms over variables" by "flattening the structure":

$$\mathbb{E}_{\Sigma,\mathcal{X}} \quad = \quad \langle \mathbb{I}_{\mathcal{T}_\Sigma\,\mathcal{X}}, \Sigma\,(\mathbb{E}_{\Sigma,\mathcal{X}})\,;\,\mathbb{A}_{\Sigma,\mathcal{X}}\rangle$$

It is easily verified that $\mathbb{V}_\Sigma$ and $\mathbb{E}_\Sigma$ are natural simulations; the remaining monad laws are equivalent to those for the standard categorical case:

Proposition 4.2. $(\mathcal{T}_\Sigma, \mathbb{V}_\Sigma, \mathbb{E}_\Sigma)$ *is an ordered monad.* □

For $R : \mathcal{A} \to \mathcal{T}_\Sigma\,\mathcal{B}$, the composition $\mathbb{I}_{\mathcal{T}_\Sigma\,\mathcal{A}} \fatsemi R$ is the application of substitution R to terms; we can derive a recursive equation for this, too:

Lemma 4.3. (*i*) $\mathbb{I} \fatsemi R = \mathcal{T}_\Sigma\,R\,;\,\mathbb{E}_{\Sigma,\mathcal{C}}$
(*ii*) $\mathbb{I} \fatsemi R = \langle R, \Sigma\,(\mathbb{I} \fatsemi R)\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle$
(*iii*) $Q \fatsemi R = Q\,;\,\langle R, \Sigma\,(\mathbb{I} \fatsemi R)\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle$

Proof. We show (i) in the first two steps of the proof for (ii):

$$\begin{array}{lll}
& \mathbb{I} \fatsemi R & \\
= & \mathbb{I}\,;\,\mathcal{T}_\Sigma\,R\,;\,\mathbb{E}_{\Sigma,\mathcal{C}} & \text{Def. } \fatsemi \\
= & \mathcal{T}_\Sigma\,R\,;\,\mathbb{E}_{\Sigma,\mathcal{C}} & \text{Identity law} \\
= & (R + \Sigma\,(\mathcal{T}_\Sigma\,R))\,;\,\langle \mathbb{I}_{\mathcal{T}_\Sigma\,\mathcal{C}}, \Sigma\,(\mathbb{E}_{\Sigma,\mathcal{C}})\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle & \text{Def. } \mathcal{T}_\Sigma\text{, Def. } \mathbb{E}_{\Sigma,_} \\
= & \langle R\,;\,\mathbb{I}_{\mathcal{T}_\Sigma\,\mathcal{C}}, \Sigma\,(\mathcal{T}_\Sigma\,R)\,;\,\Sigma\,(\mathbb{E}_{\Sigma,\mathcal{C}})\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle & \text{Sum property} \\
= & \langle R, \Sigma\,(\mathcal{T}_\Sigma\,R\,;\,\mathbb{E}_{\Sigma,\mathcal{C}})\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle & \text{Identity law} \\
= & \langle R, \Sigma\,(\mathbb{I} \fatsemi R)\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle & \text{(i)}
\end{array}$$

From this, we easily obtain (iii): $Q \fatsemi R \;=\; (Q\,;\,\mathbb{I}) \fatsemi R \;=\; Q\,;\,(\mathbb{I} \fatsemi R) \;=\; Q\,;\,\langle R, \Sigma\,(\mathbb{I} \fatsemi R)\,;\,\mathbb{A}_{\Sigma,\mathcal{C}}\rangle$. □

5. Relational substitutions

Definition 5.1. Given two variable sets $\mathcal{X}$ and $\mathcal{Y}$, a *relational* Σ*-substitution* from $\mathcal{X}$ to $\mathcal{Y}$, written $\sigma : \mathcal{X} \xrightarrow[\Sigma]{} \mathcal{Y}$, is a relation $\sigma : \mathcal{X} \to \mathcal{T}_\Sigma\,\mathcal{Y}$.

The set of all relational Σ-substitutions from $\mathcal{X}$ to $\mathcal{Y}$ is written $\mathcal{X} \xrightarrow[\Sigma]{} \mathcal{Y}$. □

The inclusion ordering $\subseteq$ on $\mathcal{X} \xrightarrow[\Sigma]{} \mathcal{Y}$, and therefore also meets and joins, are those of relations in $\mathcal{X} \to \mathcal{T}_\Sigma\,\mathcal{Y}$.

Since we have shown that the term functor $\mathcal{T}_\Sigma$ extends to an ordered monad, a relational substitution is a morphism of the Kleisli category $\mathbb{K}\,\mathcal{T}_\Sigma$, and Lemma 3.5 implies:

Proposition 5.2. *Taking variable sets as objects and relational* Σ*-substitutions between them as morphisms produces an ordered category with domain, which we denote* $\mathsf{RelSubst}_\Sigma$*, and which is defined as the Kleisli category of the ordered term monad.* □

Because of Lemma 3.6, domain minimality characterises exactly the univalent relational substitutions, and we use this to define the subcategory Subst_Σ which is equivalent to the (co-cartesian) category of standard substitutions with standard composition of substitutions:

Definition 5.3. Subst_Σ is the restriction of $\mathsf{RelSubst}_\Sigma$ to deterministic and total relational Σ-substitutions. □

Since inclusion and meets are inherited from the underlying relations, and since meets are not subject to additional requirements in lower semilattice categories, $\mathsf{RelSubst}_\Sigma$ is even a lower semilattice category.

Both Rel and $\mathcal{T}_\Sigma\,\mathsf{Rel}$ have empty relations $\emptyset$ as least elements of their homsets, but if Σ has a zero-ary function symbol, say c, then the relator $\mathcal{T}_\Sigma$ does not preserve the least element, since $(c \mapsto c) \in \mathcal{T}_\Sigma\,\emptyset$. In such cases, empty morphisms in $\mathsf{RelSubst}_\Sigma$ are not zero morphisms — for example, we have

$$\{x \mapsto c()\} \fatsemi \sigma = \{x \mapsto c()\}$$

for all relational substitutions σ, even when σ is empty.

However, if there are no zero-ary function symbols, then each term contains at least one variable, and $\mathcal{T}_\Sigma\, \emptyset = \emptyset$, so Lemma 3.8 implies:

Proposition 5.4. *If Σ has no zero-ary function symbols, then the empty relational substitutions $\emptyset_{\mathcal{X},\mathcal{Y}} : \mathcal{X} \xrightarrow[\Sigma]{} \mathcal{Y}$ are zero morphisms.* □

If f is a binary function symbol in Σ, and we consider the two relations $R = \{x \mapsto y\}$ and $S = \{x \mapsto z\}$, then the term $f(x, x)$ is associated

- by $\mathcal{T}_\Sigma\, R$ only with the term $f(y, y)$,
- by $\mathcal{T}_\Sigma\, S$ only with the term $f(z, z)$,
- by $\mathcal{T}_\Sigma\, (R \cup S)$ with the terms $f(y, y), f(y, z), f(z, y)$, and $f(z, z)$,

so in such cases, the term relator $\mathcal{T}_\Sigma$ does not even preserve binary joins.

However, if all function symbols are at most unary, then each term contains at most one variable, and the term relator $\mathcal{T}_\Sigma$ preserves all non-empty joins, in particular, $\mathcal{T}_\Sigma\, (R \cup S) = \mathcal{T}_\Sigma\, R \cup \mathcal{T}_\Sigma\, S$, and we easily obtain:

Proposition 5.5. *If Σ has no function symbols with arity greater than 1, then composition distributes over non-empty joins to its right.* □

In the narrow space between these two classes of counterexample, we essentially obtain path languages, and Lemma 3.8 implies:

Proposition 5.6. *If Σ contains only unary function symbols, then the category* $\mathsf{RelSubst}_\Sigma$ *is a complete Kleene category with domain and range.* □

6. Coherent determinisers

A unification problem is normally represented as an (injective) sequence of equations in $\mathcal{T}_\Sigma\, \mathcal{X}$

$$U = \langle l_1 = r_1, \ldots, l_n = r_n \rangle \;.$$

We will call this a "conventional unification problem".

To be able to deal with this inside our substitution categories, we first define a variable set

$$\mathcal{E}_n := \{e_1, \ldots, e_n\}$$

with pairwise distinct variables $e_1, \ldots, e_n$ serving as identifiers for the equations.

Now we can create two univalent relational substitutions collecting all the left-hand sides, respectively, all the right-hand sides ("$\#U$" denotes the cardinality of the set U):

$$\lambda_U, \rho_U \quad : \quad \mathcal{E}_{\#U} \xrightarrow[\Sigma]{} \mathcal{X} \qquad\qquad \begin{aligned} \lambda_U &:= \{i : 1..\#U \bullet e_i \mapsto l_i\} \\ \rho_U &:= \{i : 1..\#U \bullet e_i \mapsto r_i\} \end{aligned}$$

We can collect these into a two-element substitution set, or into a single relational substitution:

$$\eta_U \quad : \quad \mathcal{E}_{\#U} \xrightarrow[\Sigma]{} \mathcal{X} \qquad\qquad \eta_U \quad := \quad \lambda_U \cup \rho_U = \bigcup \mathcal{H}_U$$

The standard definition of unification specifies the most general unifier υ_U for U as an co-equaliser for λ_U and ρ_U in the category Subst_Σ, i.e., υ_U is a total and univalent substitution such that $\lambda_U \fatsemi \upsilon_U = \rho_U \fatsemi \upsilon_U$, and for any ν with $\lambda_U \fatsemi \nu = \rho_U \fatsemi \nu$ there exists a unique ϕ such that $\nu = \upsilon_U \fatsemi \phi$.

For moving this into the relational setting, we will consider *deterministic*, i.e., domain-minimal, morphisms.

Definition 6.1. In an ordered category with domain, we call a morphism M a *determiniser for* another morphism R iff $R\,;M$ is deterministic. □

Desharnais and Möller [10] show that morphisms contained in deterministic morphisms are deterministic as well. From monotonicity of composition we then immediately obtain:

Lemma 6.2. *If M is a determiniser for R, and $M' \subseteq M$, then M' is a determiniser for R, too.* □

We now explore how the concept of "most general unifier" can be transferred into the relational setting. As a first attempt, we directly transfer the co-equaliser-based definition:

Definition 6.3. In an ordered category with domain, let a class $\mathcal{D}$ of determinisers be given.
An *initial* $\mathcal{D}$*-determiniser for* a morphism R is a $\mathcal{D}$-determiniser M for R such that for every other $\mathcal{D}$-determiniser M' for R, there is exactly one morphism Φ such that $M' = M ; \Phi$. □

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "R"'] & \mathcal{B} \arrow[r, "M"'] \arrow[dr, "M'"'] & \mathcal{C} \arrow[d, dotted, "\Phi"] \\
 & & \mathcal{C}'
\end{tikzcd}$$

This choice of terminology follows the presentation of most-general unifiers by Goguen [19], and is justified by considering the category where objects are $\mathcal{D}$-determinisers for R, and morphisms from a $\mathcal{D}$-determiniser $M : \mathcal{B} \to \mathcal{C}$ to another $\mathcal{D}$-determiniser $M' : \mathcal{B} \to \mathcal{C}'$ are morphisms $F : M \to M'$ for which $M ; F = M'$.

Definition 6.4. In an ordered category with domain and range, a determiniser M for R is called *coherent* iff $\operatorname{dom} M = \operatorname{ran} R$.
□

7. Restricted residuals

Ordered categories are sufficient context to define the standard residuals for composition, which need not always exist. Where the residuals exist, we have for $X, Q : \mathcal{A} \to \mathcal{B}$ and $Y, R : \mathcal{B} \to \mathcal{C}$ and $S : \mathcal{A} \to \mathcal{C}$:

$$\begin{array}{llll}
Q ; Y \subseteq S & \Leftrightarrow & Y \subseteq (Q \backslash S) & \textit{right-residual} \\
X ; R \subseteq S & \Leftrightarrow & X \subseteq (S / R) & \textit{left-residual}
\end{array}$$

Due to the asymmetry of the Kleisli category construction, we obtain different situations for these residuals there:

Lemma 7.1. *The Kleisli category over an ordered monad has left residuals if the base category has left residuals.*

Proof. For $Q : \mathcal{A} \to \mathcal{M}\,\mathcal{B}$, $R : \mathcal{B} \to \mathcal{M}\,\mathcal{C}$, and $S : \mathcal{A} \to \mathcal{M}\,\mathcal{C}$, we have:

$$\begin{array}{lcclr}
 & Q \fatsemi R & \subseteq & S & \\
\Leftrightarrow & Q ; \mathcal{M}\,R ; \mu_{\mathcal{C}} & \subseteq & S & \text{Definition of Kleisli composition} \\
\Leftrightarrow & Q & \subseteq & S / (\mathcal{M}\,R ; \mu_{\mathcal{C}}) & \text{Left residual in base category} \quad \square
\end{array}$$

For right residuals, the situation is more complicated: In the Kleisli category over an ordered monad where the base category has residuals, we only have the following for $Q : \mathcal{A} \to \mathcal{M}\,\mathcal{B}$, $R : \mathcal{B} \to \mathcal{M}\,\mathcal{C}$, and $S : \mathcal{A} \to \mathcal{M}\,\mathcal{C}$:

$$\begin{array}{lccll}
 & Q \fatsemi R & \subseteq & S & \\
\Leftrightarrow & Q ; \mathcal{M}\,R ; \mu_{\mathcal{C}} & \subseteq & S & \text{Def. Kleisli composition} \\
\Leftrightarrow & Q ; \mathcal{M}\,R & \subseteq & S / \mu_{\mathcal{C}} & \text{Left residual in base category} \\
\Leftrightarrow & \mathcal{M}\,R & \subseteq & Q \backslash (S / \mu_{\mathcal{C}}) & \text{Right residual in base category}
\end{array}$$

Obviously, more information about $\mathcal{M}$ is required to make any progress towards right residuals.
The general, "unrestricted" residuals presented above become problematic in certain circumstances. For example, residuals of finite concrete relations are not necessarily finite again, which prevents the inclusion of residuals in the interface of software implementations of finite relations. Restricted residuals were originally introduced in [24] to remedy this particular problem.
The situation with relational substitutions is similar: If there is a relational substitution R such that

$$Q \fatsemi R = S,$$

and if Q is not surjective and R is allowed to assign values to variables outside the range of Q, then R can assign infinitely many terms to any such variable. The situation for Q is similar: if R is not total and either S is not total, or Q does not need to be univalent, then arbitrarily many associations to terms with variables outside the domain of R can be added to Q without invalidating the above equation.

Restricted residuals prevent these arbitrary effects, and therefore are defined in more cases than unrestricted residuals, at least in the ordered category of finite relational substitutions on finite sets of variables. In addition, the restricting condition establishes tighter control over the domain, respectively, range, of restricted residuals, and thus turns them into a more precise tool in a context like substitutions where domain and range issues are essential.

Definition 7.2. For morphisms $S : \mathcal{A} \to \mathcal{C}$ and $Q : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{C}$ in an ordered semigroupoid with domain and range, we define:

- the *restricted right-residual* $Q \mathbin{\backslash\!\!\!\bullet} S$: for all $Y : \mathcal{B} \to \mathcal{C}$,

$$Y \subseteq Q \mathbin{\backslash\!\!\!\bullet} S \quad \text{iff} \quad Q \,;\, Y \subseteq S \quad \text{and} \quad \operatorname{dom} Y \subseteq \operatorname{ran} Q,$$

- the *restricted left-residual* $S \mathbin{/\!\!\!\bullet} R$: for all $X : \mathcal{A} \to \mathcal{B}$,

$$X \subseteq S \mathbin{/\!\!\!\bullet} R \quad \text{iff} \quad X \,;\, R \subseteq S \quad \text{and} \quad \operatorname{ran} X \subseteq \operatorname{dom} R. \qquad \square$$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "S"] \arrow[dr, "Q"'] & & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "Q \mathbin{\backslash\!\!\!\bullet} S"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[rr, "S"] \arrow[dr, "S \mathbin{/\!\!\!\bullet} R"'] & & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "R"'] & \end{tikzcd}$$

For concrete relations, we have (using infix notation for relations):

$$\begin{aligned} y(Q \mathbin{\backslash\!\!\!\bullet} S)x \quad &\text{iff} \quad \forall x \cdot xQy \Rightarrow xSz \quad \text{and} \quad \exists x \cdot xQy \\ x(S \mathbin{/\!\!\!\bullet} R)y \quad &\text{iff} \quad \forall z \cdot yRz \Rightarrow xSz \quad \text{and} \quad \exists z \cdot yRz \end{aligned}$$

Where residuals exist, the restricted residuals can be defined using the unrestricted residuals:

Lemma 7.3. *[24, Lemma 5.2] In an ordered semigroupoid with domain and range, if the residuals $Q \backslash S$ or S/R exist, then the restricted residuals $Q \mathbin{\backslash\!\!\!\bullet} S$ respectively $S \mathbin{/\!\!\!\bullet} R$ exist, too, and we have:*

$$Q \mathbin{\backslash\!\!\!\bullet} S = \operatorname{ran} Q \,;\, (Q \backslash S), \qquad \textit{respectively}, \qquad S \mathbin{/\!\!\!\bullet} R = (S/R) \,;\, \operatorname{dom} R \qquad \square.$$

8. Membership of datatypes

An important aspect of terms is that they are containers for variables, that is, we need to consider a "membership" relation between terms and the variables they contain. An abstract treatment of membership has been established by Freyd et al. [17] using (unrestricted) left residuals in the context of allegories, but much of the material can be developed already in ordered categories.

Definition 8.1. [17] Let $\mathcal{F}$ be an endorelator. A collection of arrows $\delta_{\mathcal{A}} : \mathcal{F}\,\mathcal{A} \to \mathcal{A}$ is a *membership* of $\mathcal{F}$ if for each $R : \mathcal{A} \to \mathcal{B}$,

$$R/\delta_{\mathcal{B}} = (\mathbb{I}_{\mathcal{A}}/\delta_{\mathcal{A}}) \,;\, \mathcal{F}\,R \qquad \square$$

A related abstraction that generalises set membership are the direct powers as defined using symmetric quotients by Berghammer et al. [7] or Freyd and Scedrov [18]; in that context, element relations $\in_{\mathcal{A}} : \mathcal{A} \to \mathcal{P}\,\mathcal{A}$ are defined, and if all direct powers exist, then $\mathcal{P}$ is a power relator, and the converse of the element relation, $\in^{\smile}$, is its membership in terms of Definition 8.1.

A useful property for calculating with membership is the following:

Lemma 8.2. *[8, Exercise 6.18] $\mathcal{F}$ has membership δ iff for all $R : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{C}$, we have*

$$(R \,;\, S)/\delta_{\mathcal{C}} = (R/\delta_{\mathcal{B}}) \,;\, \mathcal{F}\,S. \qquad \square$$

Membership δ is a lax natural transformation from $\mathcal{F}$ to id, that is, we have, for $R : \mathcal{A} \to \mathcal{B}$,

$$\mathcal{F}\,R \;;\; \delta_{\mathcal{B}} \subseteq \delta_{\mathcal{A}} \,;\, R,$$

because

$$\begin{aligned} \mathcal{F}\,R \;;\; \delta_{\mathcal{B}} \subseteq \delta_{\mathcal{A}} \,;\, R &\Leftrightarrow \mathcal{F}\,R \subseteq (\delta_{\mathcal{A}} \,;\, R)/\delta_{\mathcal{B}} && \text{Def. } / \\ &\Leftrightarrow \mathcal{F}\,R \subseteq (\delta_{\mathcal{A}}/\delta_{\mathcal{A}}) \,;\, \mathcal{F}\,R && \text{Lemma 8.2} \\ &\Leftrightarrow \mathcal{F}\,R \subseteq \mathcal{F}\,R && \forall S \bullet \mathbb{I} \subseteq S/S \end{aligned}$$

As pointed out by Freyd et al. [17, Fact 1], it is in fact the largest lax natural transformation from $\mathcal{F}$ to id when assuming what they call the *identification axiom*, namely that the identity natural transformation is the largest lax natural transformation from id to id.

One easily checks that if composition distributes over some kind of joins (empty, binary, or arbitrary), then that kind of joins also preserves lax natural transformations, so we have:

Proposition 8.3. *If **C** is a complete upper semilattice category satisfying the identification axiom, and if the endorelator $\mathcal{F}$ over **C** has membership δ, then the lax natural transformations from $\mathcal{F}$ to* id *form a complete lattice with δ as its greatest element.* □

Definition 8.4. If this lattice is atomic, we call the atoms *positions*. □

Lemma 8.5. *If $\mathcal{F}$ has membership δ, and q is an idempotent subidentity on $\mathcal{F}\,\mathcal{A}$, then*

$$\mathcal{F}\,(\mathsf{ran}\,(q \,;\, \delta_{\mathcal{A}})) \quad \supseteq \quad q.$$

Proof. For every idempotent subidentity r on $\mathcal{A}$ we have:

$$\begin{aligned} r \supseteq \mathsf{ran}\,(q \,;\, \delta_{\mathcal{A}}) &\Leftrightarrow q \,;\, \delta_{\mathcal{A}} \,;\, r \supseteq q \,;\, \delta_{\mathcal{A}} && \text{Def. ran} \\ &\Leftrightarrow q \subseteq (q \,;\, \delta_{\mathcal{A}} \,;\, r)/\delta_{\mathcal{A}} && \text{Def. left res.} \\ &\Leftrightarrow q \subseteq ((q \,;\, \delta_{\mathcal{A}})/\delta_{\mathcal{A}}) \,;\, \mathcal{F}\,r && \text{Lemma 8.2} \\ &\Rightarrow q \subseteq \mathcal{F}\,r && \text{Lemma 8.6} \end{aligned}$$

□

The last step in this proof relies on the following basic range property:

Lemma 8.6. *In an ordered category with range, if q and s are idempotent subidentities and $q \subseteq R \,;\, s$, then $q \subseteq s$.*

Proof. $q = \mathsf{ran}\,q \subseteq \mathsf{ran}\,(R \,;\, s) = \mathsf{ran}\,(\mathsf{ran}\,R \,;\, s) = \mathsf{ran}\,R \,;\, s \subseteq s$. □

In the presence of appropriate greatest morphisms, we can even use membership to calculate the functor image of a subidentity:

Lemma 8.7. *If $\mathcal{F}$ has membership δ, and p is a subidentity on an object $\mathcal{A}$ for which $\top_{\mathcal{A},\mathcal{A}}$ and $\top_{\mathcal{A},\mathcal{F}\,\mathcal{A}}$ exist, then*

$$\mathcal{F}\,p = \mathsf{ran}\,((\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}}).$$

Proof. "$\subseteq$" holds because, for any subidentity r on $\mathcal{F}\,\mathcal{A}$:

$$\begin{aligned} & r \supseteq \mathsf{ran}\,((\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}}) \\ \Leftrightarrow\; & (\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}} \,;\, r \supseteq (\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}} && \text{Def. ran} \\ \Leftrightarrow\; & (\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}} \,;\, r \supseteq \top_{\mathcal{A},\mathcal{A}}/\delta_{\mathcal{A}} \,;\, \mathcal{F}\,p && \text{Lemma 8.2} \\ \Leftrightarrow\; & (\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}} \,;\, r \supseteq \top_{\mathcal{A},\mathcal{F}\,\mathcal{A}} \,;\, \mathcal{F}\,p && \forall X \bullet \top/X = \top \\ \Rightarrow\; & \mathsf{ran}\,((\top_{\mathcal{A},\mathcal{A}} \,;\, p)/\delta_{\mathcal{A}} \,;\, r) \supseteq \mathsf{ran}\,(\top_{\mathcal{A},\mathcal{F}\,\mathcal{A}} \,;\, \mathcal{F}\,p) && \text{Monotonicity of ran} \\ \Rightarrow\; & r \supseteq \mathcal{F}\,p && \mathsf{ran}\,(\top_{\mathcal{A},\mathcal{F}\,\mathcal{A}} \,;\, \mathcal{F}\,p) = \mathcal{F}\,p \end{aligned}$$

With idempotence of p together with Lemma 8.2, we have

$$(\top ; p)/\delta_{\mathcal{A}} ; \mathcal{F}\, p = (\top ; p ; p)/\delta_{\mathcal{A}} = (\top ; p)/\delta_{\mathcal{A}},$$

and therefore also $\mathcal{F}\, p \supseteq \mathsf{ran}\,((\top ; p)/\delta_{\mathcal{A}})$. □

Theorem 8.8. *The Kleisli category for an ordered monad $(\mathcal{M}, \eta, \mu)$ over an ordered category $\mathbf{C}$ with range, where $\mathcal{M}$ has a membership δ, is an ordered category pre-range, with $\mathsf{ran}_{\mathbb{K}\,\mathcal{M}}\, R = \mathsf{ran}_{\mathbf{C}}\,(R ; \delta) ; \eta$.*

If in addition $\delta_{\mathcal{B}} ; S ; \delta_{\mathcal{C}} \subseteq \mathcal{M}\, S ; \mu_{\mathcal{C}} ; \delta_{\mathcal{C}}$ for all objects $\mathcal{A}$, $\mathcal{B}$ and all morphisms $S : \mathcal{B} \to \mathcal{M}\,\mathcal{C}$, then range locality holds, too, and the Kleisli category therefore is an ordered category with range.

Proof. Every subidentity in the Kleisli category is of the shape $q ; \eta$ for a subidentity q in $\mathbf{C}$, and we have for every $R : \mathcal{A} \to \mathcal{M}\,\mathcal{B}$:

$$\begin{array}{rlr}
R \mathbin{⨾} (q ; \eta) \supseteq R & \Leftrightarrow R ; \mathcal{M}\, q \supseteq R & \text{Def. } ⨾\text{; monad law} \\
& \Rightarrow R ; \mathcal{M}\, q ; \delta_{\mathcal{B}} \supseteq R ; \delta_{\mathcal{B}} & \text{Monotonicity of composition} \\
& \Rightarrow R ; \delta_{\mathcal{B}} ; q \supseteq R ; \delta_{\mathcal{B}} & \delta \text{ lax nat. tr.} \\
& \Leftrightarrow q \supseteq \mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) & \text{Def. ran} \\
& \Rightarrow q ; \eta \supseteq \mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) ; \eta & \text{Monotonicity of composition}
\end{array}$$

This shows $\mathsf{ran}_{\mathbb{K}\,\mathcal{M}}\, R \supseteq \mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) ; \eta$. For the converse inclusion, we have:

$$\begin{array}{llll}
& \mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) ; \eta & \supseteq \; \mathsf{ran}_{\mathbb{K}\,\mathcal{M}}\, R & \\
\Leftrightarrow & R \mathbin{⨾} (\mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) ; \eta) & \supseteq \; R & \text{Def. ran in } \mathbb{K}\,\mathcal{M} \\
\Leftrightarrow & R ; \mathcal{M}\,(\mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}})) & \supseteq \; R & \text{Def. } ⨾\text{; monad law} \\
\Leftrightarrow & \mathcal{M}\,(\mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}})) & \supseteq \; \mathsf{ran}_{\mathbf{C}}\, R & \text{Def. ran in } \mathbf{C} \\
\Leftrightarrow & \mathcal{M}\,(\mathsf{ran}_{\mathbf{C}}\,(\mathsf{ran}_{\mathbf{C}}\, R ; \delta_{\mathcal{B}})) & \supseteq \; \mathsf{ran}_{\mathbf{C}}\, R & \text{locality} \\
\Leftrightarrow & \mathsf{True} & & \text{Lemma 8.5}
\end{array}$$

Assuming that, for any $S : \mathcal{B} \to \mathcal{M}\,\mathcal{C}$, furthermore $\delta_{\mathcal{B}} ; S ; \delta_{\mathcal{C}} \subseteq \mathcal{M}\, S ; \mu_{\mathcal{C}} ; \delta_{\mathcal{C}}$ holds, we obtain locality:

$$\begin{array}{rlr}
\mathsf{ran}_{\mathbb{K}\,\mathcal{M}}\,((\mathsf{ran}_{\mathbb{K}\,\mathcal{M}}\, R) \mathbin{⨾} S) & = \mathsf{ran}_{\mathbf{C}}\,(((\mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) ; \eta_{\mathcal{B}}) \mathbin{⨾} S) ; \delta_{\mathcal{C}}) ; \eta_{\mathcal{C}} & \text{Def. } \mathsf{ran}_{\mathbb{K}\,\mathcal{M}} \\
& = \mathsf{ran}_{\mathbf{C}}\,(\mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}}) ; S ; \delta_{\mathcal{C}}) ; \eta_{\mathcal{C}} & \text{Def. } ⨾\text{; monad law} \\
& = \mathsf{ran}_{\mathbf{C}}\,(R ; \delta_{\mathcal{B}} ; S ; \delta_{\mathcal{C}}) ; \eta_{\mathcal{C}} ; \eta_{\mathcal{C}} & \text{range locality in } \mathbf{C} \\
& \subseteq \mathsf{ran}_{\mathbf{C}}\,(R ; \mathcal{M}\, S ; \mu_{\mathcal{C}} ; \delta_{\mathcal{C}}) ; \eta_{\mathcal{C}} & \text{Assumption} \\
& = \mathsf{ran}_{\mathbf{C}}\,((R \mathbin{⨾} S) ; \delta_{\mathcal{C}}) ; \eta_{\mathcal{C}} & \text{Def. } ⨾ \\
& = \mathsf{ran}_{\mathbb{K}\,\mathcal{M}}\,(R \mathbin{⨾} S) & \text{Def. } \mathsf{ran}_{\mathbb{K}\,\mathcal{M}}
\end{array}$$
□

The signature relator Σ introduced after Definition 4.1 has a membership relation:

$$\begin{array}{c}
\delta_{\Sigma} \quad : \quad \Sigma\,\mathcal{S} \to \mathcal{S} \\
(f(t_1, \ldots, t_{\mathsf{arity}\,f}) \mapsto t) \in \delta_{\Sigma} \;\Leftrightarrow\; t \in \{t_1, \ldots, t_{\mathsf{arity}\,f}\}
\end{array}$$

We use this to define the *free variable relation* $\mathbb{F}_{\Sigma,\mathcal{A}} : \mathcal{T}_{\Sigma}\,\mathcal{A} \to \mathcal{A}$ by the following equation:

$$\mathbb{F}_{\Sigma,\mathcal{A}} = \langle \mathbb{I}_{\mathcal{A}}, \delta_{\Sigma,\mathcal{T}_{\Sigma}\,\mathcal{A}} ; \mathbb{F}_{\Sigma,\mathcal{A}} \rangle$$

Lemma 8.9. *$\mathcal{T}_{\Sigma}$ has membership $\delta_{\mathcal{T}_{\Sigma}} = \mathbb{F}_{\Sigma}$ which satisfies $\mathbb{V}_{\Sigma,\mathcal{A}} ; \delta_{\mathcal{T}_{\Sigma},\mathcal{A}} = \mathbb{I}_{\mathcal{A}}$.*

Proof. The last property is obvious from the definition; for the membership property for $R : \mathcal{A} \to \mathcal{B}$, we have the following "informal induction":

$$
\begin{array}{ll}
\quad R/\mathbb{F}_{\Sigma,\mathcal{B}} & \\
= (R/\mathbb{I}_{\mathcal{B}}) ; \mathbb{V}_{\Sigma,\mathcal{B}} \cup (R/(\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{B}} ; \mathbb{F}_{\Sigma,\mathcal{B}})) ; \mathbb{A}_{\Sigma,\mathcal{B}} & \text{Left residual over } \langle\_,\_\rangle \\
= R ; \mathbb{V}_{\Sigma,\mathcal{B}} \cup ((R/\mathbb{F}_{\Sigma,\mathcal{B}})/\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{B}}) ; \mathbb{A}_{\Sigma,\mathcal{B}} & \text{Identity law, residual composition} \\
= R ; \mathbb{V}_{\Sigma,\mathcal{B}} \cup (((\mathbb{I}_{\mathcal{A}}/\mathbb{F}_{\Sigma,\mathcal{A}}) ; \mathcal{T}_\Sigma R)/\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{B}}) ; \mathbb{A}_{\Sigma,\mathcal{B}} & \text{“Induction hypothesis”} \\
= R ; \mathbb{V}_{\Sigma,\mathcal{B}} \cup ((\mathbb{I}_{\mathcal{A}}/\mathbb{F}_{\Sigma,\mathcal{A}})/\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{A}}) ; \Sigma(\mathcal{T}_\Sigma R) ; \mathbb{A}_{\Sigma,\mathcal{B}} & \text{Lemma 8.2} \\
= \mathbb{V}_{\Sigma,\mathcal{A}} ; \mathcal{T}_\Sigma R \cup ((\mathbb{I}_{\mathcal{A}}/(\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{A}} ; \mathbb{F}_{\Sigma,\mathcal{A}})) ; \mathbb{A}_{\Sigma,\mathcal{A}} ; \mathcal{T}_\Sigma R & \text{Naturality (twice), residual composition} \\
= (\mathbb{V}_{\Sigma,\mathcal{A}} \cup ((\mathbb{I}_{\mathcal{A}}/(\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{A}} ; \mathbb{F}_{\Sigma,\mathcal{A}})) ; \mathbb{A}_{\Sigma,\mathcal{A}}) ; \mathcal{T}_\Sigma R & \text{Composition distributes over binary join} \\
= ((\mathbb{I}_{\mathcal{A}}/\mathbb{I}_{\mathcal{A}}) ; \mathbb{V}_{\Sigma,\mathcal{A}} \cup (\mathbb{I}_{\mathcal{A}}/(\delta_{\Sigma,\mathcal{T}_\Sigma \mathcal{A}} ; \mathbb{F}_{\Sigma,\mathcal{A}})) ; \mathbb{A}_{\Sigma,\mathcal{A}}) ; \mathcal{T}_\Sigma R & \text{Identity properties} \\
= (\mathbb{I}_{\mathcal{A}}/\mathbb{F}_{\Sigma,\mathcal{A}}) ; \mathcal{T}_\Sigma R & \text{Left residual over } \langle\_,\_\rangle\text{; Def. } \mathbb{F}_{\Sigma,\_}
\end{array}
$$
□

One easily convinces oneself that positions according to Definition 8.4 are a refinement of the traditional term position concept:

Proposition 8.10. *A position for $\mathcal{T}_\Sigma$ is an atomic lax natural transformation $\pi : \mathcal{T}_\Sigma \rightarrowtail \mathsf{id}$ with deterministic components.*

Each position corresponds to a sequence of pairs (f, i) where f is a function symbol from Σ and $i \in \{1, \dots, \mathsf{arity}\, f\}$, and for each such sequence s, we obtain a position π_s using the following inductive definition, where v is a variable (term):

$$
(v \mapsto v) \in \pi_{[\,]} \qquad\qquad (t_i \mapsto v) \in \pi_s \quad\Rightarrow\quad (f(t_1, \dots, t_{\mathsf{arity}\, f}) \mapsto v) \in \pi_{(f,i):s}
$$

This could of course be made more abstract by factoring it over the positions for Σ.

Theorem 8.8 implies that the ordered category $\mathsf{RelSubst}_\Sigma$ also has range, which identifies the free variables of the substitution; for $\sigma : \mathcal{X} \underset{\Sigma}{\to} \mathcal{Y}$, we have:

$$
\mathsf{ran}_{\mathsf{RelSubst}_\Sigma}\, \sigma = \mathsf{ran}_{\mathsf{Rel}}\, (\sigma ; \mathbb{F}_{\Sigma,\mathcal{Y}}) ; \mathbb{V}_{\Sigma,\mathcal{Y}}
$$
□

Similar to the way that domain minimality serves as a replacement of univalence, one may consider *range minimality* as a replacement of injectivity. However, this concept is not particularly useful for relational substitutions: A substitution σ is range-minimal in $\mathsf{RelSubst}_\Sigma$ iff each term in its range contains at least one free variable that no other term in its range contains. For example, $\{x \mapsto f(x, y), x \mapsto f(x, z)\}$ is range minimal, and $\{x \mapsto f(x, y), y \mapsto g(y, x)\}$ is not.

For relations, R is deterministic iff R is univalent, and if $R ; S$ is deterministic, then $\mathsf{ran}\, R ; S$ is deterministic, too, since

$$
S^{\smile} ; \mathsf{ran}\, R ; S \subseteq S^{\smile} ; R^{\smile} ; R ; S \subseteq \mathbb{I}.
$$

It is an interesting question for which more general structures the corresponding property holds; for relational substitutions it can be shown directly:

Lemma 8.11. *If $\sigma \mathbin{⨾} \tau$ is deterministic in $\mathsf{RelSubst}_\Sigma$, then so is $\mathsf{ran}\, \sigma \mathbin{⨾} \tau$.*

Proof. If $\mathsf{ran}\, \sigma \mathbin{⨾} \tau$ is not deterministic, then there are a variable x and terms $t_1 \neq t_2$ such that $\{x \mapsto t_1, x \mapsto t_2\} \subseteq \mathsf{ran}\, \sigma \mathbin{⨾} \tau$.

Since $\mathbb{V}$ is the identity in the Kleisli category, we have, with Theorem 8.8,

$$
\mathsf{ran}\, \sigma \mathbin{⨾} \tau = \mathsf{ran}\, (\sigma ; \mathbb{F}_\Sigma) ; \mathbb{V} \mathbin{⨾} \tau = \mathsf{ran}\, (\sigma ; \mathbb{F}_\Sigma) ; \tau,
$$

so there would then be a variable y and a term t such that $(y \mapsto t) \in \sigma$ and $(t \mapsto x) \in \mathbb{F}_\Sigma$. This implies $t[x\backslash t_1] \neq t[x\backslash t_2]$, which, because of $\{y \mapsto t[x\backslash t_1], y \mapsto t[x\backslash t_2]\} \subseteq \sigma \mathbin{⨾} \tau$, shows that $\sigma \mathbin{⨾} \tau$ is not univalent, either. □

9. Quotients in Dedekind categories

The property of domain-minimality (Definition 2.7) was identified by Desharnais and Möller [10] as a reasonable replacement for the concept of univalence, $R^{\smile} ; R \subseteq \mathbb{I}$, the definition of which relies on the availability of converse. Converse is a standard ingredient of formalisations of relation algebras; the simplest setting for the most important aspects of reasoning with converse are the allegories of Freyd and Scedrov [18]. In the remaining investigations, we will refer to the following variants:

Definition 9.1. An *allegory* is a lower semilattice category with involutory converse $R^{\smile}$ where the *modal rule* holds:

$$Q ; R \cap S \subseteq (Q \cap S ; R^{\smile}) ; R.$$

A *distributive allegory* is an allegory that is also an upper semilattice category with zero morphisms and satisfies distributivity laws.

A *Kleene allegory* is a distributive allegory that is also a Kleene category.

A *Dedekind category* [29,30] is a distributive allegory with residuals and top morphisms. □

Complete Dedekind categories can be considered as "heterogeneous relation algebras without complement".

Having converse and transitive closure available makes it easy to solve the problem of finding an initial coherent determiniser, see below Theorem 9.7. This theorem by itself is, however, not really relevant for our quest, since we know that substitution categories do not have a useful converse concept. Our main focus in this section is therefore to use the algebraic laws in the relevant allegories to transform the converse-based formulations into converse-free variants that will also be useful in relational substitution categories.

Definition 9.2. In an allegory, Ξ is a *partial equivalence relation (PER)* iff Ξ is symmetric and transitive, i.e., $\Xi^{\smile} \subseteq \Xi$ and $\Xi ; \Xi \subseteq \Xi$. □

Any symmetric relation is equal to its converse, and a PER is idempotent since

$$\Xi = \mathsf{dom}\,\Xi ; \Xi \subseteq \Xi ; \Xi^{\smile} ; \Xi = \Xi ; \Xi ; \Xi \subseteq \Xi ; \Xi.$$

We define quotients over PERs instead of over equivalences (which are reflexive PERs):

Definition 9.3. In an allegory, a *quotient* for a PER $\Xi : \mathcal{A} \to \mathcal{A}$ consists of an object $\mathcal{Q}$ and a morphism $\chi : \mathcal{A} \to \mathcal{Q}$ such that

$$\chi^{\smile} ; \chi = \mathbb{I}_{\mathcal{Q}} \quad \text{and} \quad \chi ; \chi^{\smile} = \Xi. \qquad \square$$

The first condition specifies that χ is univalent and surjective, while the second specifies compatibility with Ξ, and in particular $\mathsf{dom}\,\chi = \mathsf{dom}\,\Xi \; (= \Xi \cap \mathbb{I})$.

The first condition in fact allows a converse-free statement of the second, via the following more general property:

Lemma 9.4. *In an allegory, the following are equivalent conditions for a morphism R to be difunctional:*

$$R ; R^{\smile} ; R \subseteq R \quad \textit{iff} \quad R \mathbin{/\!\!\!\bullet} R = R ; R^{\smile} \quad \textit{iff} \quad R \mathbin{\bullet\!\!\!\backslash} R = R^{\smile} ; R.$$

Proof. According to definition of the restricted residual, we have $Y \subseteq R \mathbin{/\!\!\!\bullet} R$ iff

$$Y ; R \subseteq R \;\wedge\; \mathsf{ran}\,Y \subseteq \mathsf{dom}\,R, \qquad (*)$$

which is satisfied for $Y \subseteq R ; R^{\smile}$ because of difunctionality of R.

On the other hand, $(*)$ implies, together with $\mathsf{dom}\,R = \mathbb{I} \cap R ; R^{\smile}$,

$$Y = Y ; \mathsf{ran}\,Y \subseteq Y ; \mathsf{dom}\,R \subseteq Y ; R ; R^{\smile} \subseteq R ; R^{\smile}.$$

The third condition is dual to the second, since difunctionality is self-dual: $R ; R^{\smile} ; R \subseteq R \quad \Leftrightarrow \quad R^{\smile} ; R ; R^{\smile} \subseteq R^{\smile}$. □

This allows us to provide a generalised quotient definition:

Definition 9.5. In an ordered category with domain and range, a *quotient* for a morphism $\Xi : \mathcal{A} \to \mathcal{A}$ consists of an object $\mathcal{Q}$ and a deterministic surjective morphism $\chi : \mathcal{A} \to \mathcal{Q}$ such that

$$\chi \mathbin{/\!\!\!\bullet} \chi = \Xi. \qquad \square$$

This obviously coincides with Definition 9.3 for allegories, except that in ordered categories we cannot characterise PERs because symmetry is unavailable. However, we still have:

Corollary 9.6. *In an ordered category with domain and range, if χ is a quotient projection for Ξ, then Ξ is idempotent.* □

In an allegory, $R \,;\, \chi$ is deterministic iff it is univalent, i.e.:

$$\chi^{\smile} \,;\, R^{\smile} \,;\, R \,;\, \chi \subseteq \mathbb{I}$$

If χ is a deterministic determiniser for R with $\mathsf{dom}\,\chi \supseteq \mathsf{ran}\,R$, then this is equivalent to

$$R^{\smile} \,;\, R \subseteq \chi \,;\, \chi^{\smile}.$$

Since we then know that $\chi \,;\, \chi^{\smile}$ is a PER, but $R^{\smile} \,;\, R$ is not necessarily a PER, we need to take the PER closure to be able to formulate the “most general determiniser” property as an equation:

$$\chi \,;\, \chi^{\smile} = (R^{\smile} \,;\, R)^{+}$$

This assumes existence of the transitive closure, and happens to be one of the quotient conditions. This implies that surjective deterministic determinisers satisfying this equation are exactly quotient projections. They also are initial coherent determinisers:

Theorem 9.7. *In a Kleene allegory, if χ is a quotient projection for the PER $(R^{\smile} \,;\, R)^{+}$, then χ is an initial coherent determiniser for R.*

Proof. Let $\Xi := (R^{\smile} \,;\, R)^{+}$. Then we have $\chi \,;\, \chi^{\smile} = \Xi$ and $\chi^{\smile} \,;\, \chi = \mathbb{I}$ by assumption from the quotient conditions.
We have

$$\mathsf{dom}\,\chi = \mathsf{dom}\,(\chi \,;\, \chi^{\smile}) = \mathsf{dom}\,\Xi = \mathsf{ran}\,R,$$

so the discussion above shows that χ is a coherent determiniser for R.
If μ is any coherent determiniser for R, then we have $\mu = \Xi \,;\, \mu$:

$$\begin{aligned}
\mu &\subseteq \Xi \,;\, \mu && \mu \text{ is coherent for } R\\
&= (R^{\smile} \,;\, R)^{+} \,;\, \mu && \text{Def. } \Xi\\
&\subseteq (\mu \,;\, \mu^{\smile} \,;\, R^{\smile} \,;\, R)^{+} \,;\, \mu && \mu \text{ is coherent for } R\\
&= \mu \,;\, (\mu^{\smile} \,;\, R^{\smile} \,;\, R \,;\, \mu)^{+} && \text{properties of trans. closure}\\
&\subseteq \mu \,;\, (\mathsf{ran}\,\mu)^{+} && \mu \text{ is determiniser for } R\\
&= \mu
\end{aligned}$$

Therefore, μ factors over χ as $\mu = \Xi \,;\, \mu = \chi \,;\, \chi^{\smile} \,;\, \mu$:
If $\mu = \chi \,;\, \phi$ is any factoring over χ, then $\phi = \chi^{\smile} \,;\, \chi \,;\, \phi = \chi^{\smile} \,;\, \mu$, so we have unique factoring, and χ is initial. □

In this theorem, converse still occurs in the term $R^{\smile} \,;\, R$, for which we have a converse-free replacement in Lemma 9.4, but only of R is difunctional. We therefore use points to break down R into difunctional components, and use a residual property for injective Q, namely that $Q \backslash S \supseteq Q^{\smile} \,;\, S$, to obtain a converse-free variant of the remaining part of the algebraic determiniser condition:

Theorem 9.8. *In a complete Dedekind category, if the domain of $R : \mathcal{A} \to \mathcal{B}$ is generated by its points, i.e., $\mathsf{dom}\,R = \bigcup\{p \mid \mathsf{isPoint}\,p \wedge p \subseteq \mathsf{dom}\,R\}$, then we have:*

$$\bigcup\{Q \mid Q \subseteq R \;\bullet\; Q \backslash R\} = R^{\smile} \,;\, R$$

Proof. For “$\subseteq$”, we assume $Q \subseteq R$ and obtain:

$$\begin{aligned}
Y \subseteq Q \backslash R &\Leftrightarrow && \mathsf{dom}\,Y \subseteq \mathsf{ran}\,Q \;\wedge\; Q \,;\, Y \subseteq R && \text{Def. restr. residual}\\
&\Rightarrow && \mathsf{dom}\,Y \subseteq \mathsf{ran}\,Q \;\wedge\; Q^{\smile} \,;\, Q \,;\, Y \subseteq Q^{\smile} \,;\, R && \text{Monotonicity of composition}\\
&\Rightarrow && \mathsf{dom}\,Y \subseteq \mathsf{ran}\,Q \;\wedge\; \mathsf{ran}\,Q \,;\, Y \subseteq Q^{\smile} \,;\, R && \text{Def. ran}\\
&\Leftrightarrow && \mathsf{dom}\,Y \subseteq \mathsf{ran}\,Q \;\wedge\; Y \subseteq Q^{\smile} \,;\, R && \mathsf{dom}\,Y \subseteq \mathsf{ran}\,Q\\
&\Rightarrow && Y \subseteq R^{\smile} \,;\, R && Q \subseteq R
\end{aligned}$$

For "$\supseteq$", if p is a point (Definition 2.9) contained in $\mathrm{dom}\,R$, then $p\,;R$ is injective and therefore difunctional, so we have:

$$(p\,;R)\mathbin{\backslash\!\!\bullet}R = (p\,;R)\mathbin{\backslash\!\!\bullet}(p\,;R) = R^{\smile}\,;p^{\smile}\,;p\,;R = R^{\smile}\,;p\,;R$$

Then we obtain, with distributivity over arbitrary joins:

$$\begin{aligned}
\bigcup\{Q \mid Q \subseteq R \;\bullet\; Q\mathbin{\backslash\!\!\bullet}R)\} \;&\supseteq\; \bigcup\{p \mid \mathsf{isPoint}(p) \wedge p \subseteq \mathsf{dom}\,R \;\bullet\; (p\,;R)\mathbin{\backslash\!\!\bullet}R\}\\
&=\; \bigcup\{p \mid \mathsf{isPoint}(p) \wedge p \subseteq \mathsf{dom}\,R \;\bullet\; R^{\smile}\,;p\,;R\}\\
&=\; R^{\smile}\,;R.
\end{aligned}$$

□

This suggests the following "converse-free" concepts:

Definition 9.9. In an ordered category with domain and range, we define for any morphism $R : \mathcal{A} \to \mathcal{B}$ (conditional on existence of the joins and of all the restricted residuals)

- its *domain spread* $R^{\rhd} : \mathcal{A} \to \mathcal{A}$, with $R^{\rhd} := \bigcup\{p \mid \mathsf{isPoint}(p) \wedge p \subseteq \mathsf{ran}\,R \;\bullet\; R\mathbin{/\!\!\bullet}(R\,;p)\}$,
- its *range spread* $R^{\lhd} : \mathcal{B} \to \mathcal{B}$, with $R^{\lhd} := \bigcup\{p \mid \mathsf{isPoint}(p) \wedge p \subseteq \mathsf{dom}\,R \;\bullet\; (p\,;R)\mathbin{\backslash\!\!\bullet}R\}$,

and several abbreviations:

$$\begin{array}{lll}
R^{\rhd\!\!\rhd} := (R^{\rhd})^{+} & R^{\unrhd\!\!\rhd} := (R^{\rhd})^{*} & \\
R^{\lhd\!\!\lhd} := (R^{\lhd})^{+} & R^{\lhd\!\!\unlhd} := (R^{\lhd})^{*} & R^{\boxtimes} := R\,;R^{\lhd\!\!\unlhd}
\end{array}$$

□

From the proof of Theorem 9.8, we immediately obtain:

Corollary 9.10. *In a complete Dedekind category, we have $R^{\rhd} = R\,;R^{\smile}$ and $R^{\lhd} = R^{\smile}\,;R$, and $R^{\boxtimes}$ is the difunctional closure of R.* □

In their Dedekind category meaning, $R^{\boxtimes}$, $R^{\lhd\!\!\unlhd}$ and $R^{\unrhd\!\!\rhd}$ have originally been introduced in [21], and used for a relational characterisation of pushouts, so it is not surprising that they should also play a rôle in determinisation, which we introduced as generalisation of co-equalisers.

Furthermore, Theorem 9.7 gives us, together with Definition 9.5, a converse-free determiniser result, although only in the converse-laden Dedekind category context:

Corollary 9.11. *In a complete Dedekind category, a quotient projection for $R^{\lhd}$ is an initial coherent determiniser for R.* □

As we shall see below in Section 10, this does, unfortunately, not hold in the ordered category of relational substitutions. However, the solution we will adopt there will employ range spread, and generalise it again essentially according to the same principle we used for the definition of domain and range spread.

10. Determinisation of relational substitutions

According to Lemma 7.1, $\mathsf{RelSubst}_{\Sigma}$ has left residuals. However, right residuals, which correspond to matching, do not always exist. As an example, consider $R_1 := \{x \mapsto f(x,x), x \mapsto f(y,y)\}$. Then for each mapping X among the four mappings of the variable set $\{x,y\}$ to itself, we have:

$$R_1\,;(X\,;\mathbb{V}) \subseteq R_1,$$

but none of the resulting larger joins still satisfies this, so there is no right residual $R_1\backslash R_1$, and also no restricted right residual $R_1\mathbin{\backslash\!\!\bullet}R_1$.

Nevertheless, this is determinisable, and the substitution $\chi_1 := \{x \mapsto z, y \mapsto z\} : \{x,y\} \to \{z\}$ is deterministic and surjective, and satisfies

$$\chi_1\mathbin{/\!\!\bullet}\chi_1 = R_1^{\lhd\!\!\unlhd} = R_1^{\lhd} = \{x \mapsto x, x \mapsto y, y \mapsto x, y \mapsto y\}.$$

This does not always work; consider $R_2 := \{z \mapsto f(x, g(x)), z \mapsto f(g(a), y)\}$, which has initial coherent determiniser $\chi_2 := \{x \mapsto g(a), y \mapsto g(g(a))\}$, but

$$\begin{aligned}\chi_2 \between \chi_2 &= \{x \mapsto x, x \mapsto g(a), y \mapsto y, y \mapsto g(x), y \mapsto g(g(a))\}\\ R_2^{\lhd} = R_2^{\ntrianglelefteq} &= \{x \mapsto x, y \mapsto y\} = \mathbb{I}_{\{x,y\}}\end{aligned}$$

The problem in the construction of $R_2^{\lhd}$ is obviously that partial solutions for one position conflict even with the identity solution of the other position. So, since $p\,;R_2$ always selects full terms from the range of R_2, no non-trivial matching candidates $(p\,;R_2)\between R_2$ arise.

We can however take positions into account during construction of extended spreads — note that in allegories, $R/\mu_{\mathcal{B}} = R\,;\breve{\mu_{\mathcal{B}}}$:

Definition 10.1. Given, over an ordered category **C**, an ordered monad $(\mathcal{M}, , \eta, \mu)$ with membership δ and with position set Π, we define for a morphism $R : \mathcal{A} \to \mathcal{M}\,\mathcal{B}$ in **C** and a position $\pi : \Pi$ the *restriction of R to π*:

$$(R \downharpoonleft \pi) : \mathcal{A} \to \mathcal{M}\,\mathcal{B} \qquad R \downharpoonleft \pi := (R/\mu_{\mathcal{B}})\,;\pi_{\mathcal{M}\,\mathcal{B}}$$

Then we define the *extended range spread* $R^{\unlhd} : \mathcal{B} \to \mathcal{M}\,\mathcal{B}$ of R conditional on the existence of all constituent range spreads, which are to be calculated in the Kleisli category $\mathbb{K}\,\mathcal{M}$:

$$R^{\unlhd} := \bigcup\{\pi : \Pi \bullet (R \downharpoonleft \pi)^{\lhd}\}$$

We also define an abbreviation *in the Kleisli category*: $R^{\ntrianglelefteq} := (R^{\unlhd})^{+}$. □

(Extended domain spread can be defined analogously.)

$R \downharpoonleft \pi$ composes the substitution R with extraction of subterms from position π. For the example R_2 above, there are only five positions π for which $R_2 \downharpoonleft \pi$ is non-empty; for each of these, we show also its range spread:

$$\begin{aligned}
&R_2 \downharpoonleft [] &&= R_2 &&(R_2 \downharpoonleft [])^{\lhd} &&= \{x \mapsto x, y \mapsto y\}\\
&R_2 \downharpoonleft [(f,1)] &&= \{z \mapsto x, z \mapsto g(a)\} &&(R_2 \downharpoonleft [(f,1)])^{\lhd} &&= \{x \mapsto x, x \mapsto g(a)\}\\
&R_2 \downharpoonleft [(f,1),(g,1)] &&= \{z \mapsto a\} &&(R_2 \downharpoonleft [(f,1),(g,1)])^{\lhd} &&= \emptyset\\
&R_2 \downharpoonleft [(f,2)] &&= \{z \mapsto g(x), z \mapsto y\} &&(R_2 \downharpoonleft [(f,2)])^{\lhd} &&= \{x \mapsto x, y \mapsto y, y \mapsto g(x)\}\\
&R_2 \downharpoonleft [(f,2),(g,1)] &&= \{z \mapsto x\} &&(R_2 \downharpoonleft [(f,2),(g,1)])^{\lhd} &&= \{x \mapsto x\}
\end{aligned}$$

The join over these individual range spreads produces, after transitive closure, exactly the morphism for which χ_2 is a quotient projection:

$$\begin{aligned}R_2^{\unlhd} &= \{x \mapsto x, x \mapsto g(a), y \mapsto y, y \mapsto g(x)\}\\ R_2^{\ntrianglelefteq} &= \{x \mapsto x, x \mapsto g(a), y \mapsto y, y \mapsto g(x), y \mapsto g(g(a))\} \;=\; \chi_2 \between \chi_2\end{aligned}$$

Note that for mismatching symbols, the restricted residuals do not exist, not even in the case of constants: The restricted residual

$$\{x \mapsto a\}\between\{x \mapsto b\}$$

does not exist, although $\{x \mapsto a\} : \{x\} \to \{\}$, since $\perp_{\{\},\mathcal{T}\{\}} = \mathbb{V}_{\{\}}$ is an identity in the Kleisli category, and therefore $\{x \mapsto a\} \fatsemi \perp_{\{\},\mathcal{T}\{\}} = \{x \mapsto a\}$.

More interesting is the case of occur-check violations: Consider

$$R_3 := \{z \mapsto f(y,x), z \mapsto f(x, g(y))\}.$$

This yields

$$R_3^{\unlhd} = \{x \mapsto x, x \mapsto y, x \mapsto g(y), y \mapsto x, y \mapsto y\},$$

so $R_3^{\ntrianglelefteq}$ is infinite (but with finite range $\{x,y\}$), and has no quotient projection as long as only finite terms are allowed.

If we were to allow rational terms, R_3 would have the determiniser

$$\chi_3 := \{x \mapsto g^{\infty}, y \mapsto g^{\infty}\},$$

and this would be a quotient projection for $R_3^{\ntrianglelefteq}$.

Another option would be to switch to a non-complete category of finite substitutions; then the transitive closure $R_3^{\trianglelefteq} = (R_3^{\triangleleft})^+$ would not exist.

In any case, this failure can be detected in finite time (note that we are dealing with concrete relations here) since

$$R_3^{\triangleleft} ; \delta$$

has a cycle leading through an edge induced by a non-empty position, i.e.,

$$R_3^{\triangleleft} ; \mathsf{ran}\, \mathbb{A} ; \delta ; (R_3^{\triangleleft} ; \mathbb{V}^{\smile})^*$$

has a cycle.

In summary, the extended range spread $R^{\trianglelefteq}$ decomposes a unification problem in the same way as conventional algorithms, but aggregates the partial results in a way that formally makes the unifier a quotient projection:

Fact 10.2. In $\mathsf{RelSubst}_\Sigma$, if $R^{\trianglelefteq}$ exists, then a quotient projection for $R^{\trianglelefteq}$ is an initial coherent determiniser for R. □

Determining the precise conditions under which a quotient projection for $R^{\trianglelefteq}$ is an initial coherent determiniser for the morphism R in the Kleisli category over an ordered monad with membership and positions remains open for future work.

11. Related work

Rydeheard and Burstall [31] and Goguen [19] (who used the dual setting) pointed out that unification corresponds to determining co-equalisers in the Kleisli category of the term monad.

Instead of using an ordered monad, relational substitutions can also be obtained as morphisms in the Kleisli category of the composition of the powerset monad with the term monad. Monad composition does work under certain conditions, several of these were developed by Jones and Duponcheel [20], one of them being the presence of a "distributive law" originally proposed by Beck [4], or equivalently a "swapper" natural transformation, which Eklund et al. [15] use to show that the composition $\mathcal{T}_\Sigma ; \mathbb{P}$ of the term functor with the powerset functor can be extended to a monad, too. Note that arbitrary monads cannot necessarily be composed to a new monad as shown by Jones and Duponcheel [20]. The string rewriting approach of that proof is explicitly elaborated by Kozen [26] to produce a general tool for verifying monad compositions and re-prove the monadicity of $\mathcal{T}_\Sigma ; \mathbb{P}$. Eklund et al. [14] replaced the standard powerset monad $\mathbb{P}$ with L-fuzzy powerset monads.

Eklund and Helgesson [16] use a "partially ordered monad" concept restricted to endofunctors on Set and show that under certain conditions the resulting Kleisli category is a Kleene category. These conditions make intrinsic use of Set structure and establish the result by guaranteeing that the Kleisli category is a complete upper semilattice category.

Where Eklund et al. [13] proceed to use the composed monad for unification, they consider equations consisting of *two* relational substitutions, just like previous work on unification in the categorical context.

The use of relators for datatype definition has been pioneered by Backhouse et al. [3], see also Backhouse and Hoogendijk [1]; a recent overview is the text by Bird and de Moor [8]. Much of the material there can be used to present signature functors and term monads in a more formal way, and to perform fully formal calculations.

12. Conclusion

For a relatively general kind of relational categories, we introduced the concept of *determiniser* which enables treatment of unification problems represented as a *single* relational morphism.

In a previous paper [23], we showed that this determiniser concept successfully deals with such one-morphism formulations of unification problems, relying on the same initiality concept as traditional unification.

The current paper was motivated by the desire to replace this initiality condition, which is a typical categorical universal property, with a local algebraic property in the flavour of relation-algebraic characterisations of direct sums and products, or, in particular quotients.

By considering PERs instead of equivalence relations and by using restricted residuals, we managed to avoid the ubiquitous totality and surjectivity constraints present in [23]. As a result, we obtained a generalised quotient concept for arbitrary locally ordered categories with domain and range, that applies both to conventional quotients and to initial determinisers, i.e., essentially to unification.

Expressing the required kernel of the quotient projection for initial determinisers in a similarly general way proved more challenging, but we managed to produce a palatable concept of "extended range spread" for this purpose by using a position concept derived directly from abstract membership of datatypes.

For future work, it would be nice to be able to characterise the use of conditions in the extended spread definition without exposing the Kleisli category structure. Also, moving to the full Backhouse–Bird–de Moor calculational approach to recursive datatypes should enable us to derive matching and unification algorithms in a calculational way.

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References

[1] R. Backhouse, P. Hoogendijk, Elements of a relational theory of datatypes, in: B. Möller, H. Partsch, S. Schuman (Eds.), Formal Program Development. Proc. IFIP TC2/WG 2.1 State of the Art Seminar, Rio de Janeiro, January 1992, LNCS, vol. 755, 1992, pp. 7–42.
[2] R.C. Backhouse, Constructive lattice theory, October 1993. <http://www.cs.nott.ac.uk/∼rcb/papers/abstract.html#isos>.
[3] R.C. Backhouse, P.J. deBruin, G. Malcom, E. Voermans, J. vander Woude, Relational catamorphisms, in: B. Möller (Ed.), Constructing Programs From Specifications. IFIP WG 2.1,, North-Holland, 1991, pp. 319–371.
[4] J. Beck, Distributive laws, in: H. Appelgate, B. Eckmann (Eds.), Seminar on Triples and Categorical Homology Theory, ETH, 1966–1967, Lect. Notes in Math., vol. 80, Springer, 1969, pp. 119–140.
[5] R. Berghammer, A.M. Haeberer, G. Schmidt, P.A.S. Veloso, Comparing two different approaches to products in abstract relation algebra, in: M. Nivat, C. Rattray, T. Rus, G. Scollo (Eds.), AMAST '93. Workshops in Computing, Springer, 1994, pp. 167–176.
[6] R. Berghammer, A. Jaoua, B. Möller (Eds.), Relations and Kleene Algebra in Computer Science – 11th International Conference on Relational Methods in Computer Science, and Sixth International Conference on Applications of Kleene Algebra, RelMiCS/AKA 2009, Doha, Qatar, November 1–5, 2009. ProceedingsLNCS, vol. 5827, Springer, 2009.
[7] R. Berghammer, G. Schmidt, H. Zierer, Symmetric quotients and domain constructions, Inform. Process. Lett. 33 (3) (1989) 163–168.
[8] R.S. Bird, O. de Moor, Algebra of programming, International Series in Computer Science, vol. 100, Prentice Hall, 1997.
[9] J. Desharnais, P. Jipsen, G. Struth, Domain and antidomain semigroups, in: R. Berghammer et al. (Eds.), 2009, pp. 73–87.
[10] J. Desharnais, B. Möller, Characterizing determinacy in Kleene algebras, Inform. Sci. 139 (2001) 253–273.
[11] J. Desharnais, B. Möller, G. Struth, Kleene algebra with domain, ACM Trans. Comput. Logic 7 (4) (2006) 798–833.
[12] H. Doornbos, N. van Gasteren, R. Backhouse, Programs and datatypes, in: C. Brink, W. Kahl, G. Schmidt (Eds.), Relational Methods in Computer Science, Advances in Computing Science, Springer, Wien, New York, 1997, pp. 150–165.(Ch. 10)
[13] P. Eklund, M.A. Galán, J. Medina, M. OjedaAciego, A. Valverde, A categorical approach to unification of generalised terms, Electron. Notes Comput. Sci. 66 (5) (2002) 41–51.(Special Issue: UNCL'2002, Unification in Non-Classical Logics (ICALP 2002 Satellite Workshop))
[14] P. Eklund, M.A. Galán, J. Medina, M. OjedaAciego, A. Valverde, Set functors, *I*-fuzzy set categories, and generalized terms, Comput. Math. Appl. 43 (6–7) (2002) 693–705.
[15] P. Eklund, M.A. Galán, M. Ojeda-Aciego, A. Valverde, Set functors and generalised terms, in: Proc. IPMU 2000, Eighth Information Processing and Management of Uncertainty in Knowledge-based Systems Conference, vol. III, 2000, pp. 1595–1599.
[16] P. Eklund, R. Helgesson, Composing partially ordered monads, in: R. Berghammer et al. (Eds.), 2009, pp. 88–102.
[17] P. Freyd, P. Hoogendijk, O. de Moor, Membership of datatypes, unpublished manuscript, December 1993 (see also Bird and de Moor, 1997, Sect. 6.5).
[18] P.J. Freyd, A. Scedrov, Categories, allegories, North-Holland Mathematical Library, vol. 39, North-Holland, Amsterdam, 1990.
[19] J.A. Goguen, What is unification?, in: H. Aït-Kaci, M. Nivat (Eds.), Resolution of Equations in Algebraic Structures. 1: Algebraic Techniques, Academic Press, Boston, 1989, pp. 217–261.
[20] M.P. Jones, L. Duponcheel, Composing monads, Research Report YALEU/DCS/RR-1004, Yale University, New Haven, CT, USA, December 1993.
[21] W. Kahl, A relation-algebraic approach to graph structure transformation, Habil. Thesis, Fakultät für Informatik, Univ. der Bundeswehr München, Techn. Report 2002-03, 2001. <http://sqrl.mcmaster.ca/∼kahl/Publications/RelRew/>.
[22] W. Kahl, Refactoring heterogeneous relation algebras around ordered categories and converse, J. Relational Methods Comput. Sci. 1 (2004) 277–313.
[23] W. Kahl, Determinisation of relational substitutions in ordered categories with domain, in: R. Berghammer, B. Möller, G. Struth (Eds.), Relations and Kleene-Algebra in Computer Science, RelMiCS/AKA 2008, LNCS, vol. 4988, Springer, 2008, pp. 243–258.
[24] W. Kahl, Relational semigroupoids: Abstract relation-algebraic interfaces for finite relations between infinite types, J. Logic Algebraic Program. 76 (1) (2008) 60–89.
[25] Y. Kawahara, Notes on the universality of relational functors, Mem. Fac. Sci. Kyushu Univ. Ser. A 27 (2) (1973) 275–289.
[26] D. Kozen, Natural transformations as rewrite rules and monad composition, Tech. Rep. TR2004-1942, Computer Science Department, Cornell University, July 2004.
[27] F.W. Lawvere, Functorial semantics of algebraic theories, Proc. Natl. Acad. Sci. USA 50 (1963) 869–872.
[28] B. Möller, Kleene getting lazy, Sci. Comput. Programming 65 (2007) 195–214.
[29] J.-P. Olivier, D. Serrato, Catégories de Dedekind. Morphismes dans les catégories de Schröder, C. R. Acad. Sci. Paris Ser. A–B 290 (1980) 939–941.
[30] J.-P. Olivier, D. Serrato, Squares and rectangles in relation categories – three cases: Semilattice, distributive lattice and boolean non-unitary, Fuzzy Sets and Systems 72 (1995) 167–178.
[31] D. Rydeheard, R. Burstall, A categorical unification algorithm, in: Proc. Summer Workshop on Category Theory and Computer Programming 1985, LNCS vol. 240, Springer (1986) 493–505.
[32] G. Schmidt, T. Ströhlein, Relation algebras – concept of points and representability, Discrete Math. 54 (1985) 83–92.
[33] J.M. Spivey, The Z Notation: A Reference Manual, second ed., Prentice Hall International Series in Computer Science, Prentice Hall, 1992 (out of print). <http://spivey.oriel.ox.ac.uk/∼mike/zrm/>.